Journal of Statistics and Computer Science
Vol. 1, No. 1, 2022, pp. 77-98

URL : www.arfjournals.com
https://DOI: 10.47509 /JSCS.2022.v01i01.06

A Sequential Design for Estimating the Product of Two Non Simultaneously Zero Means

Zohra Benkamra[a], Mekki Terbeche[a] and Mounir Tlemcani[a*]
[a]*Departement de Mathematiques, Université des Sciences et Technologie d'Oran Mohamed Boudiaf, USTO-MB, Algerie. E-mail: mounir.tlemcani@gmail.com*

Received April 12, 2022 • Revised June 7, 2022 • Accepted June 10, 2022 • Published July 15, 2022

Abstract: The aim is to estimate the product of two arbitrary means of independent populations possibly with dierent and unspecied distributions. We give a three-stage sequential design which allocates the number of observations from each population such that the sample variance can be lowered up to second order terms to its true lower bound. A new and interesting logarithmic first stage length is proposed and the design is shown to be second order asymptotically optimal assuming at least one mean is non zero.

Keywords: Nonlinear Sequential design, Second order, Asymptotic optimality, Three-stage, Sampling, product of means.

2020 MSC: 62L10, 62L12

To cite this article

Zohra Benkamra, Mekki Terbeche & Mounir Tlemcani. (2022). A Sequential Design for Estimating the Product of Two Non Simultaneously Zero Means. *Journal of Statistics and Computer Science*. Vol. 1, No. 1, pp. 11-32. *https://DOI: 10.47509 /JSCS.2022.v01i01.06*

1. Introduction

Effective estimation of the ratio and/or product of means relates to a more general problem of estimating a nonlinear function of parameters, Maji *et al.* (2018), Raubenheimer and van der Merwe (2012), Singh and Karpe (2009). A product of means like Bernoulli proportions, Benkamra *et al.* (2015, 2013), Rekab and Wu (2015), Page (1990, 1987), Poisson rates Kim (2006), Raubenheimer and van der Merwe (2012), is naturally involved in series/parallel systems and software reliability. Product of normal means, Berger and Bernardo (1989), Southwood (1978), Sun and Ye (1999, 1995), Yfantis and Flatman (1991), is used in area estimation problems based on measurements of length and width or in ecological systems such as the study of insect populations, Southwood (1978).

When the allocation schemes are constrained to cost or risk per unit observation, it is possible to design sequential procedures that are ecient, at least asymptotically for large samples, Benkamra *et al.* (2015, 2013), Rekab and Wu (2015), Zheng *et al.* (1998), Page (1990, 1987), Woodroofe and Hardwick (1990).

Assume that for $i = 1, ..., n$; a random variable Xi whose distribution is unknown but is observable from a population $(\mathcal{P}_i)$ such that moments exist up to a sucient order. If the total sample size T is fixed, then the problem of mestimating a function of the means $\mu_i = \mathbb{E}(X_i)$ such as a sum or a product can be processed by sequential allocation. In such procedures, the unknowns are the sample sizes m_i (the number of observations from population $(\mathcal{P}_i)$) such that the sample variance can be made as small as possible. Asymptotic optimality can be considered when the total sample size T is large, *i.e.* $T \to +\infty$.

In this paper we give a three-stage sequential design for estimating a product two of arbitrary means with possibly unspecied distributions assuming at least one of them is non zero. This includes namely the case of two normal means with unknown and unequal variances as in inference procedures encountered in Behrens-Fisher-type problems Chaturvedi *et al.* (2020), Liu and Wang (2007). Recently, in Benkamra *et al.* (2015), a linear version of this scheme was used for a particular case of estimating a product of Bernoulli proportions. The scheme proposed here is based on the nonlinear expression of the exact variance and it is shown to be asymptotically second order optimal in the sense that the sample variance approaches the exact lower bound at a speed $o(1/T^{\gamma})$ for all $\gamma < 2$. When the means are not zero, it is well known that optimality conditions are reduced to a proportionality relation between sample sizes m_i and the coecients of variation ci of the corresponding populations, Page (1990). However, when a population $(\mathcal{P}_i)$ has a mean $\mu_i = 0$ (or close to 0), its coeficient of variation c_i is either undened or will approach innity and its estimation is therefore sensitive to small changes in the sample mean. So the proposed scheme is based rather on the inverse of these coecients of variation in order to keep nonlinear terms signicant in the ratio $m_i = T$. Particular works in the literature can be found for specied families of distributions such as the exponential family in a Bayesian framework where posterior means are not zero as in reliability estimation problems when the parameters are Bernoulli proportions subject to Beta-priors, see Benkamra *et al.* (2013), Page (1987) and references therein. Throughout this work, we will restrict the study to the case of two non simultaneously zero means 1 and 2. The populations are assumed independent and the distributions may be unspecied. The main result of this work relies in a new choice of the first stage length such as $\mathcal{O}(\log(T))$ in order to obtain second order asymptotic optimality of the design, instead of powers of T as it has been used in the literature cited above. Therefore, when one of the two means is zero (for example $\mu_1 = 0$) then the partition $m_1 + m_2 = T$ will balance most of sampling units towards the non-zero mean population, *i.e.* looks as $m_1 = o(T)$ and $m_2 = \mathcal{O}(T)$, unlike the case where both means are non-zero which led to sample sizes all proportional to T, *i.e.* $m_1 = \mathcal{O}(T)$ and $m_2 = \mathcal{O}(T)$. More precisely, the use of nonlinear terms in the expression of exact lower bound of the sample variance is more than necessary to take into account such sampling behavior when one of the two means may be zero. In Section 2, we present the problem setting for two parameters. An exact lower bound of the sample variance is derived in Section 3. In the fourth section, we propose a first stage length $L = \log T$, which to our knowledge is new with respect to discussions found on optimal few-stage designs, Hardwick and Stout (2002) and other references therein. Asymptotic properties of the allocation numbers m_1, m_2 are derived for large total sample size T. In Section 5, a three-stage procedure is presented similar to Benkamra *et al.* (2015) where the authors have considered a product of Bernoulli proportions. The new scheme is based on the nonlinear expression of the exact lower bound of the sample variance. We then give the main result on the asymptotic efficiency up to second order of the sequential design even in the case of one zero mean. In the sixth section, we present by Monte-Carlo simulation some validation of the theoretical results obtained in both homogeneous and heterogeneous couple of populations that we have specified in order to run the simulations. An example with two simultaneously zero normal means is also presented in this section. We observe that the allocation rule agrees with the asymptotic optimality condition. Finally, proofs of Lemmas and the main theorem are given in Appendix, Section 7.

In all what follows, standard notations for asymptotic comparison in probability are used as follows: $f = o(g)$, $f = \mathcal{O}(g)$, respectively, as $T \to \infty$, means that f is dominated by g asymptotically with probability one, f is dominated and subjected to g asymptotically with probability one, as $T \to \infty$.

2. The Problem Setting

Assume that two random variables X_1, X_2 are conditionally independent, non degenerate and have nite moments of order at least up to 4, that is for $i = 1, 2$, $\mathbb{E}(|X_i|^p) < +\infty$ for all $p \in [1, 4[$, in particular,

$$\mathbb{E}[X_i] = \mu_i < +\infty,\ 0 < \mathbb{V}[X_i] = \sigma_i^2 < +\infty \tag{2.1}$$

Suppose that a total and fixed number of observations T is allowed such that in each population $(\mathcal{P}_i)$ a sample size m_i is allocated for estimating the mean μ_i. Our aim in this paper is to construct an allocation rule for sample sizes m_1, m_2 under the constraint $m_1 + m_2 = T$ fixed, in order to estimate efficiently the product

$$\pi = \mu_1\mu_2.$$

We know that the following empirical estimator for is unbiased and convergent,

$$\hat{\theta} = \bar{X}_{1,m_1}\bar{X}_{2,m_2},$$

where $\bar{X}_{i,m_i}$ is the sample mean of population $(\mathcal{P}_i)$,

$$\bar{X}_{i,m_i} = \frac{S_{im_i}}{m_i} = \frac{\sum_{j=1}^{m_i} X_{ij}}{m_i}$$

Since such estimators are unbiased then the risk under quadratic loss reduces to the sample variance $R(\hat{\theta}) = \mathbb{V}(\hat{\theta})$, here the probability measure for (X_1, X_2) is dened by the product measure. Thus, assuming independence within and across populations, the sample variance

$$\mathbb{V}(\hat{\theta}) = \mathbb{E}\left(\bar{X}_{1,m_1}^2\right)\mathbb{E}\left(\bar{X}_{2,m_2}^2\right) - \mathbb{E}^2\left(\bar{X}_{1,m_1}\right)\mathbb{E}^2\left(\bar{X}_{2,m_2}\right) \tag{2.2}$$

can be straight forwardly (see next section) written as a function of allocation numbers m_1, m_2 and consequently can be lowered by choosing convenient designs or partitions of the total sample size $T = m_1 + m_2$ when T is fixed.

3. Exact Lower Bound

In this section, we give a suitable expression of the exact lower bound of the sample variance. We have

$$\mathbb{E}\left(\bar{X}_{i,m_i}^2\right) = \mathbb{V}\left(\bar{X}_{i,m_i}\right) + \mathbb{E}^2\left(\bar{X}_{i,m_i}\right), i = 1, 2,$$

and by relation (2.1), we obtain by (2.2) the following expression of the sample variance,

$$\mathbb{V}(\hat{\theta}) = \frac{\mu_2^2\sigma_1^2}{m_1} + \frac{\mu_1^2\sigma_2^2}{m_2} + \frac{\sigma_1^2\sigma_2^2}{m_1 m_2}. \tag{3.1}$$

By a Lagrange-type identity one can write with the help of the constraint condition $m_1 + m_2 = T$,

$$\mathbb{V}(\hat{\theta}) = Q + E(m_1, m_2),$$

where

$$Q = \frac{1}{T}\left(\sigma_2\sqrt{\mu_1^2 + \frac{\sigma_1^2}{T}} + \sigma_1\sqrt{\mu_2^2 + \frac{\sigma_2^2}{T}}\right), \tag{3.2}$$

$$E(m_1, m_2) = \frac{\left(m_1\sigma_1\sqrt{\mu_2^2 + \frac{\sigma_2^2}{T}} - m_2\sigma_2\sqrt{\mu_1^2 + \frac{\sigma_1^2}{T}}\right)^2}{T m_1 m_2} \tag{3.3}$$

The quantity Q does not depend on m_1, m_2 and represents the exact lower bound for $\mathbb{V}(\hat{\theta})$ under the constraint $m_1 + m_2 = T$. The second term $E = E(m_1, m_2) \geq 0$ is called the excess of variance uncured by the design (m_1, m_2), such that $m_1 + m_2 = T$. When $E = 0$, *i.e.*, when

$$\frac{m_1}{m_2} = \frac{\sigma_2\sqrt{\mu_1^2 + \frac{\sigma_1^2}{T}}}{\sigma_1\sqrt{\mu_2^2 + \frac{\sigma_2^2}{T}}}, \tag{3.4}$$

the design is called optimal, assuming that m_1 and m_2 are treated as continuous variables. In addition, it is nonlinear since the proportionality ratio in (3.4) is a nonlinear function of the sample size T, unlike to what is found in the literature where this ratio is rather constant so that m_1, m_2 become linear functions of T, Benkamra *et al*. (2015), Page (1990). More specically, this non linearity is needed when one or both of the two means may be zero. When 1 = 2 = 0, we remark that relation (3.4) implies that the optimal sampling design is balanced, *i.e.* $m_1 = m_2 = T/2$, yielding $E(m_1, m_2) = 0$ and this even when σ_1 is not equal to σ_2. In particular, this balanced or equal allocation rule happens when the populations have equal parameters $\mu_1 = \mu_2$ and $\sigma_1 = \sigma_2$.

4. First Stage Sequential Design

In what follows, we will assume that at least one mean is non zero, for example $\mu_2 \neq 0$. Remark that the exact optimality conditions (3.4) can be written equivalently as

$$m_1 = T\frac{\sigma_2\sqrt{\mu_1^2+\frac{\sigma_1^2}{T}}}{\sigma_2\sqrt{\mu_1^2+\frac{\sigma_1^2}{T}}+\sigma_2\sqrt{\mu_2^2+\frac{\sigma_2^2}{T}}},\ m_2 = T - m_1, \tag{4.1}$$

which is the starting idea to construct sequential schemes to solve approximately the discrete optimization problem when the parameters μ_1, σ_1 and μ_2, σ_2 are unknowns. Let us point that the cases $\{\mu_1 \neq 0, \mu_2 \neq 0\}$ and $\{\mu_1 = 0, \mu_2 \neq 0\}$ give respectively rise to two dierent behaviors of the sample size m_1, such as $m_1 = \mathcal{O}(T)$ in the first case and $m_1 = o(T)$ in the second one, with probability one as $T \to \infty$.

Let us denote by $(\hat{\mu}_i, \hat{\sigma}_i^2)$ empirical estimators of (μ_i, σ_1^2), $i = 1, 2$, obtained by a first stage sampling with sample size $L \leq T = 4$ for example in each population $(\mathcal{P}_i)$. In the literature, L is generally a function of T that must satisfy some conditions such as $L \to \infty$ and $L = o(T)$ at the same time, as $T \to \infty$. For example, a standard candidate is $L = \mathcal{O}(T)$ with 0 β < 1 fixed, or more usually $L = \mathcal{O}(\sqrt{T})$. Remark that when the product $\mu_1\mu_2$ may be zero, such condition on first stage length L above is not sucient to reach second order efficiency in the sequential design.

The following lemma is based on the strong law of large numbers (S.L.L.N) of Marcinkiewicz which gives a rate of convergence in the (S.L.L.N) of Kolmogorov-Khintchine, Stout (1974).

Lemma 1. Assume $\mu_2 \neq 0$, $\mathbb{E}(|X_i|^p) < \infty$ for all $p \in [1, 4[$, $i = 1, 2$, and dene $\hat{m}_i$ by

$$\hat{m}_2 = T - \hat{m}_i,\ \frac{\hat{m}_1}{T} = \frac{\hat{\sigma}_2\sqrt{\hat{\mu}_1^2+\frac{\hat{\sigma}_1^2}{T}}}{\hat{\sigma}_2\sqrt{\hat{\mu}_1^2+\frac{\hat{\sigma}_1^2}{T}}+\hat{\sigma}_2\sqrt{\hat{\mu}_2^2+\frac{\hat{\sigma}_2^2}{T}}}, \tag{4.2}$$

where $\hat{\mu}_i, \hat{\sigma}_i^2$ are empirical estimators of μ_i, σ_i^2 based on same sample size $L \le T$, *i.e.*,

$$\hat{\mu}_i = \overline{X}_{i,L}, \hat{\sigma}_i^2 = \overline{X}_{i,L}^2 - \hat{\mu}_i^2,$$

such that $L \to +\infty$, as $T \to +\infty$. Then, if $\mu_1 \ne 0$,

$$\frac{\hat{m}_1}{T} = \frac{\sigma_2\sqrt{\mu_1^2 + \frac{\sigma_1^2}{T}}}{\sigma_2\sqrt{\mu_1^2 + \frac{\sigma_1^2}{T}} + \sigma_1\sqrt{\mu_2^2 + \frac{\sigma_2^2}{T}}} + o\left(\frac{1}{L^q}\right), \text{ as } T \to +\infty, \tag{4.3}$$

with probability one, for any q; $0 \le q < 1/2$. Moreover, if $\mu_1 = 0$ then (4.3) remains true if in addition $L = o(T)$.

Remark that when $\mu_1 = 0$, Equation (4.3) reduces simply under the condition $L = o(T)$ to

$$\frac{\hat{m}_1}{T} = o\left(\frac{1}{L^q}\right), \text{ as } T \to +\infty. \tag{4.4}$$

In sequential design, the partition numbers m_1, m_2 have generally to grow both at same order of the total sample size $T = m_1 + m_2$ as as $T \to +\infty$,, *i.e.* $m_i/T = \mathcal{O}(1)$. However, in the particular case when $\mu_1 = 0$ and $\mu_2 \ne 0$ in Lemma (1), one can observe that the sample size $\hat{m}_1$ as given by relation (4.4) is no longer increasing at the same rate as T. We propose in the following lemma a new first stage length, namely a logarithmic function of T, in order to control this growing rate relatively to powers less than one of the total sample size T.

Lemma 2. Assume $\mu_2 \ne 0$ and let $\hat{m}_1$, $\hat{m}_2$ dened as in Lemma (1) associated to a first stage length $L = \log T$, then $\hat{m}_2 = \mathcal{O}(T)$, and

(i) if $\mu_1 \ne 0$, $\hat{m}_1 = \mathcal{O}(T)$,

(ii) if $\mu_1 = 0$, $\hat{m}_1 = o(T)$ and for all β, $0 \le \beta < 1$, $\frac{\hat{m}_1}{T^\beta} \to +\infty$ with probability one, as $T \to +\infty$.

Underline the fact that the particular case when $\mu_1 = \mu_2 = 0$ is not covered by this Lemma. However, when $\mu_1 = 0$ and $\mu_2 \ne 0$, the second point (ii) of this lemma shows that even though $\hat{m}_1$ remains dominated by T for large samples, it nevertheless grows faster than any power of T less than one. This is actually a main advantage in this case in order to keep the proof of second order asymptotic optimality rightful.

5. A Sequential Three-stage Design for Estimating A Product of Non-Simultaneously Zero Means

Assume in this section that 1 and 2 are not simultaneously zero, for example $\mu \ne 0$. We will denote by $[x]$ the nearest integer to the real number x. Dealing with two parameters, one can start a design associated to a total sample size equals $[T/2]$ instead of T and according to Lemma (2) a first stage length $L = [\log(T/2)]$ such that we have always $2L \leq [T/2]$ and $L \to +\infty$ as $T \to +\infty$. The sample sizes $\hat{m}_i$ proposed by the first stage procedure (see Section 4 highlighting here that $\hat{m}_1 + \hat{m}_2 = [T/2]$) behave as described by Lemma (2). With these sample sizes in hand, one can update in the second stage the moments estimators in order to improve their accuracy. In a third and final stage, these rened moments estimators are used to update $\hat{m}_i$ according to the complete sample size T similarly as proposed in Lemma (1)-Eq.(4.2).

The three-stage design

1st stage:

Sample $L \leq [T/4]$ from each population and evaluate the moments estimators

$$\hat{\mu}_i = \overline{X}_{i,L} = \frac{\sum_{j=1}^{L} X_{i,j}}{L}, \hat{\sigma}_i^2 = \frac{\sum_{j=1}^{L} X_{i,j}^2}{L} - \hat{\mu}_i^2, i = 1, 2. \tag{5.1}$$

2nd stage:

Sample $\hat{m}_i - L$ more observations in population $\mathcal{P}_i$, such that

$$\hat{m}_2 = \left[\frac{T}{2}\right] - \hat{m}_1, \hat{m}_1 = \min\left\{\max\left\{L, [S_{L,L}]\right\}, \left[\frac{T}{2}\right] - L\right\}, \tag{5.2}$$

where

$$S_{L,L} = \frac{T}{2} \frac{\hat{\sigma}_2 \sqrt{\hat{\mu}_1^2 + \frac{\hat{\sigma}_2^2}{T}}}{\hat{\sigma}_2 \sqrt{\hat{\mu}_1^2 + \frac{\hat{\sigma}_2^2}{T}} + \hat{\sigma}_1 \sqrt{\hat{\mu}_2^2 + \frac{\hat{\sigma}_2^2}{T}}} \tag{5.3}$$

and update the moments estimators by

$$\hat{\mu}_i = \overline{X}_{i,\hat{m}_i} = \frac{\sum_{j=1}^{\hat{m}_i} X_{i,j}}{\hat{m}_i}, \hat{\sigma}_i^2 = \frac{\sum_{j=1}^{L} X_{i,j}^2}{L} - \hat{\mu}_i^2, i = 1, 2. \tag{5.4}$$

3rd stage:

Sample $m_i - \hat{m}_i$ more in population $\mathcal{P}_i$, such that

$$m_2 = T - m_1, m_1 = \min\{\max\{\hat{m}_1, [S_{\hat{m}_1,\hat{m}_2}]\}, T - \hat{m}_2\}, \tag{5.5}$$

where

$$S_{\hat{m}_1,\hat{m}_2} = T\frac{\hat{\sigma}_2\sqrt{\hat{\mu}_1^2 + \frac{\hat{\sigma}_2^2}{T}}}{\hat{\sigma}_2\sqrt{\hat{\mu}_1^2 + \frac{\hat{\sigma}_2^2}{T}} + \hat{\sigma}_1\sqrt{\hat{\mu}_2^2 + \frac{\hat{\sigma}_2^2}{T}}}, \tag{5.6}$$

update the empirical means by

$$\overline{X}_{i,m_i} = \frac{\sum_{j=1}^{m_i} X_{i,j}}{m_i}, \; i = 1, 2,$$

and estimate the product of means $\mu_1\mu_2$ by

$$\hat{\theta} = \overline{X}_{1,m_1}\overline{X}_{2,m_2}.$$

Theorem 1. Under the same assumptions of Lemma 1, let m_1, m_2 given by the three-stage design with a first stage length $L = [\log(T/2)]$, then the excess of variance of $\hat{\theta} = \overline{X}_{1,m_1}\overline{X}_{2,m_2}$ incurred by the sampling design $\{m_1, m_2\}$ satisfies

$$E(m_1, m_2) \le o\left(\frac{1}{T^\gamma}\right) \tag{5.7}$$

as $T \to +\infty$, with probability one, for all γ, $0 \le \gamma < 2$.

It should be pointed in this theorem that contrary to what is used in the literature $L = \mathcal{O}(\sqrt{T})$ for first stage length, this choice is no longer sucient when one of the two means may be zero, as far as the corresponding sample size becomes dominated by T. Hence, the choice of a logarithmic first stage length imposes itself by the result (ii) of Lemma (2). Moreover, the case of two zero means $\mu_1 = \mu_2 = 0$ is not yet covered by this theoretical result. Despite this, the simulation shows promising results.

6. Monte-Carlo Simulation

In this section, we will show dierent experiments in order to validate the theoretical results obtained above. We will consider a total sample size T increasing from 2 to 100 by step of $\Delta T = 1$. All the Monte-Carlo (MC) simulations were done with 5000 replication when sampling, Robert and Casella (2004). We will present respectively homogeneous and heterogeneous couples of distributions. Balanced and unbalanced cases will correspond to the situations where the allocation numbers m_1 and m_2 are equal or unequal, respectively.

In order to validate second order asymptotic optimality, plots of the decay speed $T^\gamma(m_1, m_2)$ are given for a power $\gamma = 1.99$ where $E(m_1, m_2) = \mathbb{V}ar(\hat{\theta}) - Q$ denotes the excess of variance. This power was chosen less than 2 because the convergence result given by Theorem 1 is not valid for $\gamma = 2$. This fact is due to the rate of convergence given by the strong law of large numbers which is upper limited by the central limit theorem. However, because of the centrallimit theorem and since we consider in this paper asymptotic estimation with large samples, several examples are given with normal means.

6.1. Experiments with two means with Common Distribution

6.1.1. Product of two Bernoulli means

We consider a simple case of two proportions p_1 and p_2 in]0, 1[. For example, with $p_1 = 0.5$ and $p_2 = 0.1$ in unbalanced case and $p_1 = p_2 = 0.1$ in the balanced situation. Figure 1 (A-1: unbalanced and B-1: balanced) shows T^γ times the excess of variance over the true lower bound incurred by 3-stage design with $\gamma = 1.99$, *i.e.*, $T^{1.99}(\mathbb{V}ar(\hat{\theta}) - Q)$ where Q is the exact lower given by (3.2). In Figure 1 - A-2, the ratio $\frac{m_1}{m_2}$ converges eectively with probability one to the expected proportion (unbalanced case) given by (3.4), *i.e.*

$$\frac{\sigma_2 p_1}{\sigma_1 p_2} = \frac{\sqrt{p_2(1-p_2)p_1}}{\sqrt{p_1(1-p_1)p_2}} = 3$$

and in Figure 1 - B-2 we can observe namely that this ratio tends to 1 which means that $m_1 = m_2 \to T/2$, *i.e.* asymptotically the design becomes balanced.

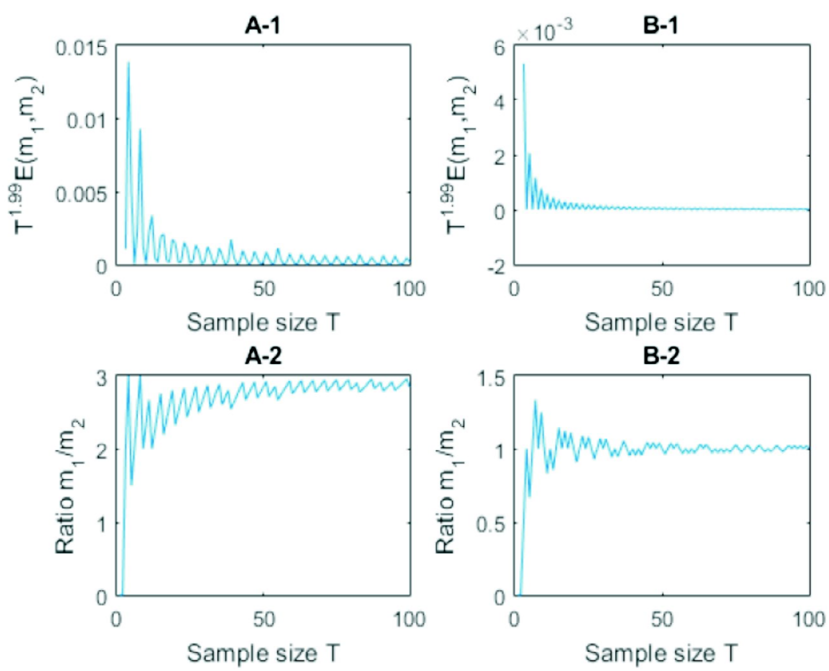

Figure 1. Three-stage design - Bernoulli proportions - A. Unbalanced case $p_1 = 0.5$, $p_2 = 0.1$, and B. Balanced case $p_1 = p_2 = 0.1$.

6.1.2. Product of two Poisson Rates

We consider now the case of two Poisson rates λ_1 and λ_2 in $]0, +\infty[$. For example, with $\lambda_1 = 1$ and $\lambda_2 = 2$ in unbalanced case and $\lambda_1 = \lambda_2 = 2$ in the balanced situation. Figure 2 (A-1: unbalanced and B-1. balanced) shows T^{γ} times the excess of variance over the true lower bound incurred by the 3-stage design with $\gamma = 1.99$, *i.e.*, $T^{1.99}E(m_1, m_2) = T^{1.99}(\mathbb{V}ar(\hat{\theta}) - Q)$ where Q is the exact lower bound. We see in Figure 2 - A-2 that the ratio $\frac{m_1}{m_2}$ converges with probability one to the expected proportion

$$\frac{\sigma_2|\mu_1|}{\sigma_1|\mu_2|} = \frac{\sqrt{\lambda_2}\lambda_1}{\sqrt{\lambda_1}\lambda_2} = 0.7071$$

and in Figure 2 - B-2 we can observe namely that this ratio tends to 1 which means that asymptotically the design becomes balanced.

6.1.3. Product of two uniform means

We consider here the case of two uniform means θ_1 and θ_2 in $]0, +\infty[$. For example, $\theta_1 = 2.5$ and $\theta_2 = 3$ in unbalanced case and $\theta_1 = \theta_2 = 3$ in the balanced situation. Figure 3 (A-1: unbalanced and B-1: balanced) shows T^{γ} times the excess of variance over the true lower

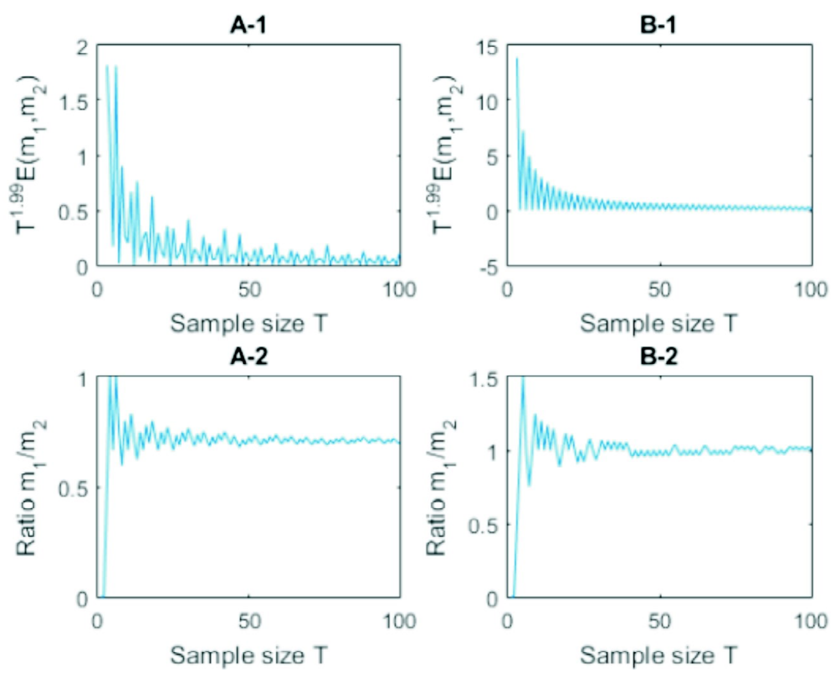

Figure 2. Three-stage design - Poisson rates - A: Unbalanced case $\lambda_1 = 1$, $\lambda_2 = 2$, and B: Balanced case $\lambda_1 = \lambda_2 = 2$.

bound incurred by 3-stage design with $\gamma = 1.99$. In Figure 3 - A-2, the ratio $\frac{m_1}{m_2}$ converges with probability one to

$$\frac{\sigma_2\theta_1}{\sigma_1\theta_2} = \frac{\frac{\theta_2}{\sqrt{3}}\theta_1}{\frac{\theta_1}{\sqrt{3}}\theta_2} = 1 \tag{6.1}$$

and in Figure 3 - B-2 we can also observe that this ratio tends to 1. Specically in the case of uniform populations, the balanced design is the best for any θ_1, θ_2 non zero as indicated by (6.1). Namely, this same result will happen for any distribution whose standard deviation is proportional to its mean, for example in the case of an exponential distribution $\mathcal{E}(\theta)$ parametrized by $\theta = 1/\lambda$ since its variance equals θ^2. In fact, the ratio $\frac{m_1}{m_2}$ converges with probability one to

$$\frac{\sigma_2\theta_1}{\sigma_1\theta_2} = \frac{\theta_2\theta_1}{\theta_1\theta_2} = 1 \tag{6.2}$$

6.1.4. Product of two non zero normal means - $\mu_1\mu_2 \neq 0$

Consider two normal means μ_1 and μ_2 in $]-\infty, +\infty[$ such that $\mu_1\mu_2 \neq 0$. A: Unbalanced case $\mu_1 = -1$, $\mu_2 = 1.5$, $\sigma_1 = 1$, $\sigma_2 = 2.5$, and B: Balanced case $\mu_1 = \mu_2 = 1.5$ and $\sigma_1 = \mu_2 = 2.5$. Figure 4 (A-1: unbalanced and B-1: balanced) shows T^γ times the excess of variance over the true lower bound incurred by 3-stage design with $\gamma = 1.99$. In Figure 4 - A-2 the ratio $\frac{m_1}{m_2}$ converges with probability one to the expected proportion

$$\frac{\sigma_2|\mu_1|}{\sigma_1|\mu_2|} = 1.6667,$$

and in Figure 4 - B-2 we can observe namely that this ratio tends to 1 which means that asymptotically the design becomes balanced.

6.1.5. Product of two normal means - case $\mu_1\mu_2 = 0$

Consider two normal means μ_1 and μ_2 in $]-\infty, +\infty[$ such that $\mu_1\mu_2 = 0$. A: Unbalanced case $\mu_1 = 0$, $\mu_2 = 1.5$, $\sigma_1 = 1$, $\sigma_2 = 2.5$, and B: Balanced case $\mu_1 = \mu_2 = 0$ and standard deviations $\sigma_1 = 1$, $\mu_2 = 2.5$. Figure 5 (A-1: unbalanced and B-1: balanced) shows T times the excess of variance over the true lower bound incurred by 3-stage design with $\gamma = 1.99$. In Figure 5 - A-2, the ratio $\frac{m_1}{m_2}$ converges with probability one to zero while in Figure 5 - B-2 we can observe that this ratio tends to 1 which means that $m_1 = m_2 \to T/2$, *i.e.* asymptotically the design becomes balanced.

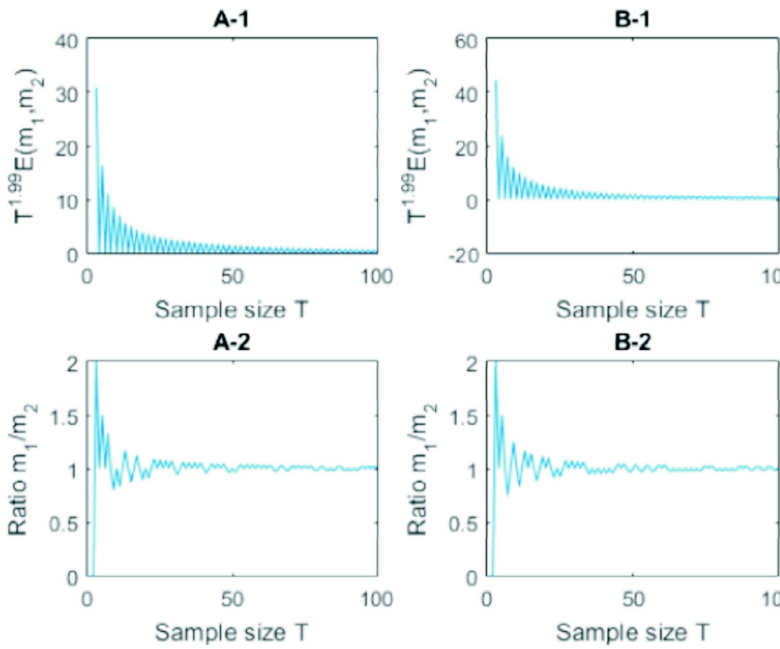

Figure 3. Three-stage design - Uniform means - A: Unbalanced case 1 = 2.5, 2 = 3, and B: Balanced case $\theta_1 = \theta_2 = 3$.

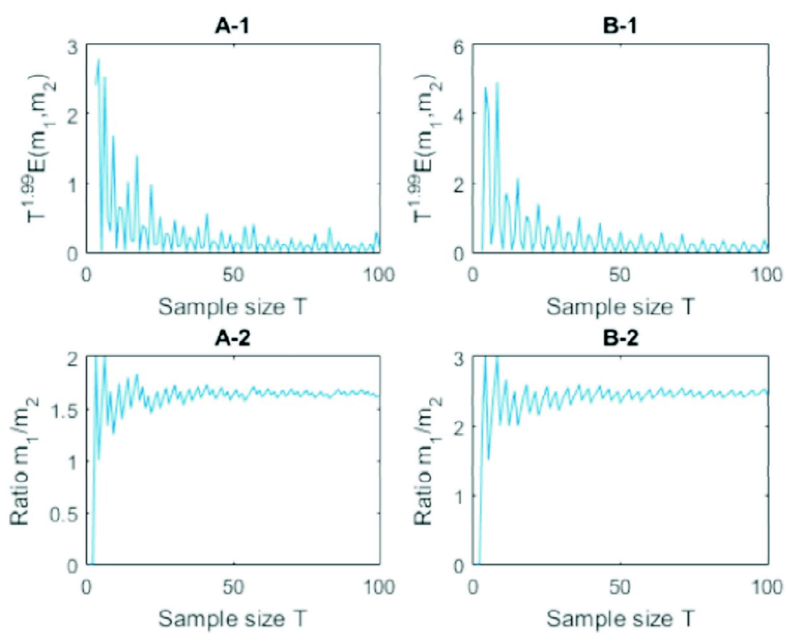

Figure 4. Three-stage design - Normal means - A: Unbalanced case $\mu_1 = -1$, $\mu_2 = 1.5$, $\sigma_1 = 1$, $\sigma_2 = 2.5$, and B. Balanced case $\mu_1 = \mu_2 = 1.5$ and $\sigma_1 = \sigma_2 = 2.5$.

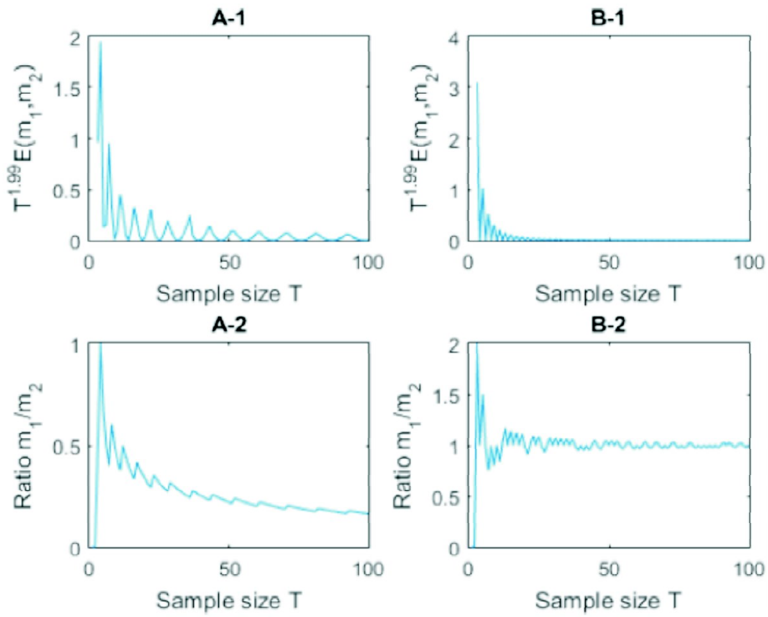

Figure 5. Three-stage design - Normal means with $\mu_1\mu_2 = 0$ - A: Unbalanced case $\mu_1 = 0$, $\mu_2 = 1.5$, $\sigma_1 = 1$, $\sigma_2 = 2.5$, and B: Balanced case $\mu_1 = \mu_2 = 0$, $\sigma_1 = 1$, $\mu_2 = 2.5$.

Remark that numerically the case $\mu_1 = \mu_2 = 0$ behave right according to the exact (balanced) design $m_1 = m_2 \to T/2$, as shown in Figure 5 B-2. Unfortunately, theoretical proof of convergence with probability one in the case $\mu_1 = \mu_2 = 0$ remains not achieved in this framework. However, the simulation has shown that the choice $L = \mathcal{O}(\log T)$ for first stage sample size, intensively tested by the authors, gives in many other and dierent situations the true solution of the optimization problem, *i.e.*, the balanced partition $m_1 = m_2$ with a second order excess of variance incurred by the design, see Figure 5 B-1. We have also tested the case of non simultaneously zero normal means to highlight the fact that a standard choice of first stage length $L = \mathcal{O}(\sqrt{T})$ results effectively in a loss of second order convergence of the three-stage design as shown in Figure 6. The experiment is same as above with the two normal populations $\mathcal{N}(0, 1)$ and $\mathcal{N}(1, 0.1)$. We observe only a first order rate of convergence. $T^{\beta}(m_1, m_2)$ is not decreasing with T for $\beta = 1.99$, it is bounded for $\beta = 1.5$ and decreasing for $\beta = 1$. This fact conrms the inadequacy of standard first stage length when one of the two means may be zero.

6.2. Experiments with two means of Dierent Distributions

6.2.1. Product of Normal by Bernoulli means

In this experiment, we consider the product $\theta = \mu p$ where μ is the mean of a normal population $\mathcal{N}(\mu, \sigma)$ and $p \in]0, 1[$ is a Bernoulli parameter. Figures 7 A- and 7 B- show MC simulation

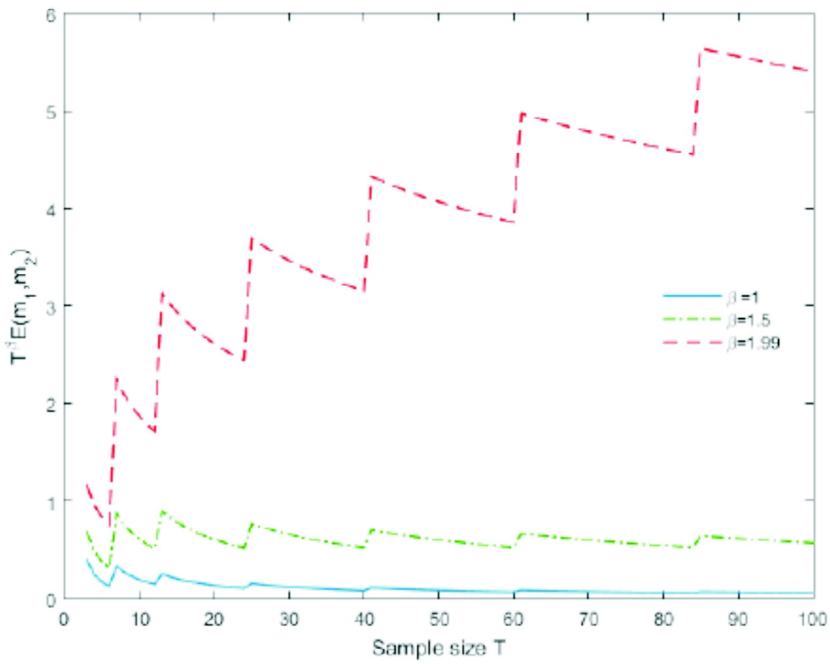

Figure 6. Loss of second order convergence relative to the choice of first stage length $L = \sqrt{(T/2)}$. Normal means with $\mu_1 = 0$, $\mu_2 = 1$, $\sigma_1 = 1$, $\mu_2 = 0.1$. The three-stage design is only first order w.r.t. *T*.

for test cases $\{\mu = 1.5, p = 2.5, p = 0.3\}$ as example for $\theta \neq 0$ and $\{\mu = 0, \sigma = 2.5, p = 0.3\}$ as example for $\theta = 0$, respectively. The excess of variance behaves in both cases correctly, see Figure 7 A-1, B-1. Likewise for the ratio m_1/m_2. As expected, $m_1/m_2 \to 0$ when $\theta = 0$, see Figure 7 B-2.

6.2.2. Product of Normal by Exponential means

In this experiment, we consider the product $\theta = \mu.1/\lambda$ where μ is the mean of a normal population $\mathcal{N}(\mu, \sigma)$ and $1/\lambda \in]0, +\infty[$ the exponential mean. Figures 8 A- and 8 B- show MC simulation for test cases $\{\mu = 1.5, \sigma = 2.5, 1/\lambda = 0.3\}$ as example for $\theta \neq 0$ and $\{\mu = 0, \sigma = 2.5, 1/\lambda = 0.3\}$ as example for $\theta = 0$, respectively. The excess of variance behaves in both cases correctly, see Figure 8 A-1 B-1. Likewise for the ratio m_1/m_2. As expected, $m_1/m_2 \to 0$ when $\theta = 0$, see Figure 8 B-2.

6.2.3. Product of Exponential by Uniform means

This is a particular example where the allocation will be independent of the parameters of the distributions. In fact, let us consider the product $\Theta = 1/\lambda.\theta$ where λ is the scale parameter of an exponential distribution $\mathcal{E}(\lambda)$ and θ the mean of a uniform variable $\mathcal{U}(2\theta)$. Here, $\mu_1 = \sigma_1 = 1/\lambda$ for $\mathcal{E}(\lambda)$ and $\mu_2 = \theta$, $\sigma_2 = \theta/\sqrt{3}$ for $\mathcal{U}(2\theta)$. Hence forth, the exact optimality condition 3.4 gives asymptotically, for large samples ($T \to +\infty$):

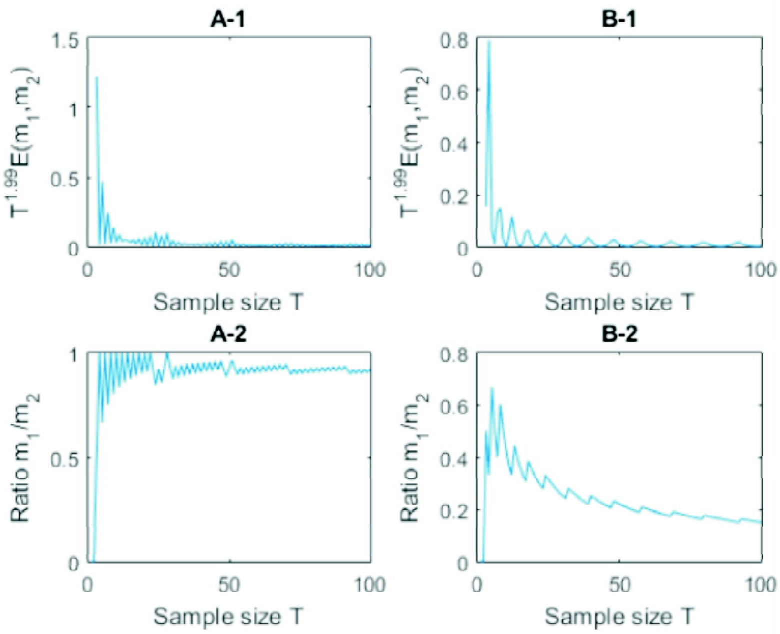

Figure 7. Three-stage design - Normal by Bernoulli means with - A: case $\mu_1 = \mu = 1.5$, $\mu_2 = p = 0.3$, $\sigma_1 = \sigma = 2.5$, and B. case $\mu_1 = \mu = 0$, $\mu_2 = p = 0.3$, $\sigma_1 = \sigma = 2.5$.

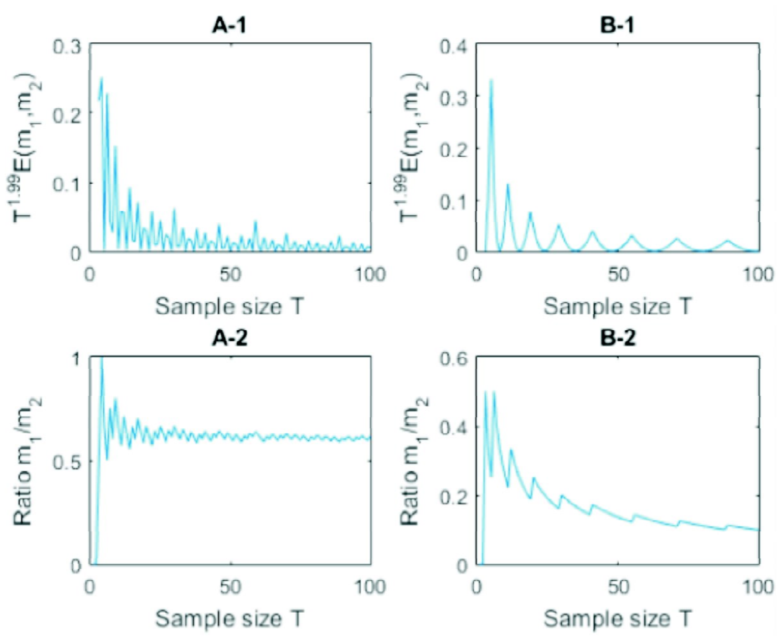

Figure 8. Three-stage design - Normal by Exponential means with - A: case $\mu_1 = \mu = 1.5$, $\mu_2 = 1/\lambda = 0.3$, $\sigma_1 = \sigma = 2.5$, and B. case $\mu_1 = \mu = 0$, $\mu_2 = 1/\lambda = 0.3$, $\sigma_1 = \sigma = 2.5$.

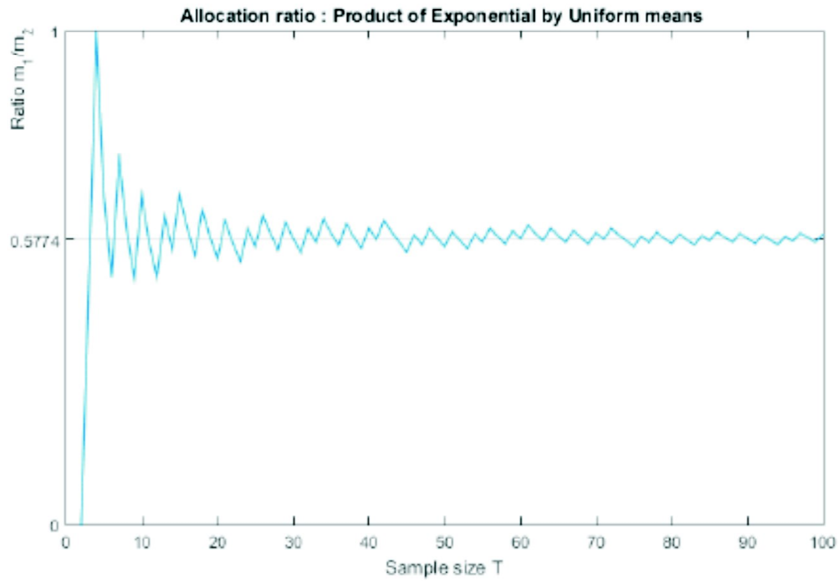

Figure 9. Three-stage design - Exponential by uniform means with λ = 10 and θ = 1. Asymptotically, the allocation ratio tends to the exact value 0.5774.

$$\frac{m_1}{m_2} = \frac{\sigma_2\sqrt{\mu_1^2 + \dfrac{\sigma_1^2}{T}}}{\sigma_1\sqrt{\mu_2^2 + \dfrac{\sigma_2^2}{T}}} \to \frac{\sigma_2\mu_1}{\sigma_1\mu_2} = 1/\sqrt{3} \simeq 0.5774$$

The allocation ratio 0.5774 is well validated in Figure 9 and this result can be observed for several other values of the parameters in many other experiences that are not shown here.

6.2.4. Product of Normal by Uniform means

As last example, we consider the product $\theta = \mu.\theta_2$ where μ is the mean of a normal population $\mathcal{N}(\mu, \sigma)$ and $\theta_2 \in]0, +\infty[$ a uniform mean.

Figures 10 A- and 10 B- show MC simulation for test cases $\{\mu = 1.5, \sigma = 2.5, \theta_2 = 1.5\}$ as example for $\theta \neq 0$ and $\{\mu = 0, \sigma = 2.5, \theta_2 = 1.5\}$ as example for $\theta = 0$, respectively. The excess of variance behaves in both cases as desired, see Figure 10 A-1 B-1. As expected, $m_1/m_2 \to 0$ when $\theta = 0$, see Figure 10 B-2.

7. Conclusion

We have proposed a new first stage length and a three-stage design for a nonlinear problem of estimating a product of two arbitrary non simultaneously zero means (possibly with unspecied distributions). The choice of a logarithmic first stage length works well and is justied theoretically except for the case where both the two means are zero. In fact, when

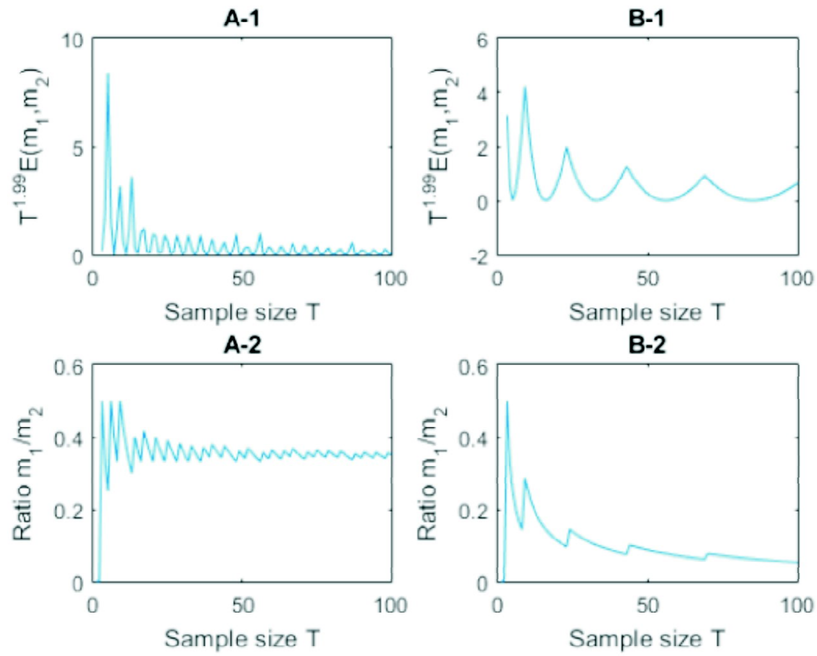

Figure 10. Three-stage design - Normal by uniform means with - A: case $\mu_1 = \mu = 1.5$, $\mu_2 = \theta_2 = 1.5$, $\sigma_1 = \sigma = 2.5$, and B: case $\mu_1 = \mu = 0$, $\mu_2 = \theta_2 = 1.5$, $\sigma_1 = \sigma = 2.5$.

one mean may be zero, the logarithmic choice is to our knowledge the only one who lets the sampling goes completely over the non zero mean population and conserves the second order convergence of the three-stage design. This is a consequence of the fact that when $T \to +\infty$, $\log T \to +\infty$ while remaining negligible compared to any positive power of T, *i.e.* $\log T/T^\beta \to 0$ for all $\beta > 0$. Unfortunately, the problem of two simultaneously zero means still not solved in this paper and needs further theoretical analysis. The general case of several arbitrary means can be considered in perspective.

Acknowledgements

The authors gratefully acknowledge and thank the reviewers for their valuable comments and recommendation to improve the last version of the draft. This work has been partially supported by the Ministry of Higher Education of Algeria, PRFU and LAAR Laboratory of USTO.

Appendix

Proof of Lemma 1

Since the random variables X_i and X_i^2 have moments of any order $p \in [1, 2[$ then the rate of convergence in the S.L.L.N of Marcinkiewicz gives, with probability one, as the sample size $L \to +\infty$, for any $q \in [0, 1/2[$,

$$\hat{\mu}_i = \bar{X}_{i,L} = \mu_i + o\left(\frac{1}{L^q}\right), \tag{7.1}$$

$$\hat{\sigma}_i^2 = \bar{X}_{i,L}^2 - \hat{\mu}_i^2 = \sigma_i^2 + o\left(\frac{1}{L^q}\right), \tag{7.2}$$

which yield, since $L \to +\infty$ as $T \to +\infty$,

$$\hat{\mu}_i^2 + \frac{\hat{\sigma}_i^2}{T} = \mu_i^2 + \frac{\sigma_i^2}{T} + \mu_i o\left(\frac{1}{L^q}\right) + o\left(\frac{1}{L^{2q}}\right) + \frac{1}{T} o\left(\frac{1}{L^q}\right). \tag{7.3}$$

Thus, if $\mu_i \neq 0$ the last two terms in the the r.h.s of (7.3) are dominated by $\mu_i o\left(\frac{1}{L^q}\right)$ such that one obtains,

$$\sqrt{\hat{\mu}_i^2 + \frac{\hat{\sigma}_i^2}{T}} = \sqrt{\mu_i^2 + \frac{\sigma_i^2}{T}} + o\left(\frac{1}{L^q}\right),$$

and if not, *i.e.* $\mu_i = 0$, then (7.3) implies

$$\sqrt{\hat{\mu}_i^2 + \frac{\hat{\sigma}_i^2}{T}} = \sqrt{\frac{\sigma_i^2}{T} + o\left(\frac{1}{L^q}\right)} = \left(\frac{1}{L^q}\right), \tag{7.4}$$

since $L^{2q} = o(T)$ in this case. Henceforth, (4.3) follows immediately and the proof is nished.

Proof of Lemma 2

Relation $\hat{m}_2 = O(T)$ follows from both (i) or (ii). Assertion (i) follows immediately from Lemma 1 - Eq. (4.3) with $m_1 \neq 0$. In the case (ii) - $m_1 = 0$, it is clear that $\hat{m}_1 = o(T)$ as $T \to +\infty$ by the same Equation (4.2), and one has from the denition of $\hat{m}_1$ given by (4.2), for any α, $0 < \alpha \leq 1$,

$$\frac{\hat{m}_1}{T^{1-\alpha}} = T^{\alpha} \frac{\hat{m}_1}{T} = \frac{\hat{\sigma}_2 \sqrt{T^{2\alpha} \hat{\mu}_1^2 + T^{2\alpha-1} \hat{\sigma}_1^2}}{\hat{\sigma}_2 \sqrt{\hat{\mu}_1^2 + \frac{\hat{\sigma}_1^2}{T}} + \hat{\sigma}_1 \sqrt{\hat{\mu}_2^2 + \frac{\hat{\sigma}_2^2}{T}}}. \tag{7.5}$$

Forget the denominator since it always converges with probability one to a positive constant as $T \to +\infty$, when $\mu_2 \neq 0$.

Thus, if $\alpha > 1/2$, $2\alpha - 1 > 0$ and since $\hat{\sigma}_1^2 \to \sigma_1^2 > 0$, then $T^{2\alpha-1} \hat{\sigma}_1^2 \to +\infty$ which yields $T^{\alpha-1} \hat{m}_1 \to +\infty$ with probability one, as $T \to +\infty$.

Otherwise, if $0 < \alpha \leq 1/2$, $2\alpha - 1 \leq 0$ then $T^{2\alpha-1} \hat{\sigma}_1^2$ becomes bounded but $T^{2\alpha} \hat{\mu}_1^2$ does not. Indeed, one has by the central limit theorem, since $\mu_1 = 0$,

$$\sqrt{L} \frac{\hat{\mu}_1}{\sigma_1} \xrightarrow{d} \mathcal{N}(0,1)$$

in distribution, as $T \to +\infty$, in one hand, which writes

$$\sqrt{\frac{L}{T^{2\alpha}}} T^{\alpha} \frac{\hat{\mu}_1}{\sigma_1} \xrightarrow{d} \mathcal{N}(0,1). \tag{7.6}$$

But since $L = \log T$ and $\alpha > 0$ then $\sqrt{\frac{L}{T^{2\alpha}}} \to 0$ as $T \to +\infty$ forcing hence the second term $T^{\alpha} \frac{\hat{\mu}_1}{\sigma_1}$ to be unbounded because if not then the hole sequence converges to zero with probability one, as $T \to +\infty$, *i.e.*

$$\sqrt{\frac{L}{T^{2\alpha}}} T^{\alpha} \frac{\hat{\mu}_1}{\sigma_1} \to 0,$$

which is in contradiction with (7.6). Henceforth, the sequence $T^{\alpha}\hat{\mu}_1$ is unbounded, with probability one, which implies that $\frac{\hat{m}_1}{T^{1-\alpha}} \to +\infty$ and achieves the proof of (ii) with $\beta = 1 - \alpha$, for all β, $0 \le \beta < 1$.

Proof of Theorem 1.

By assumption on $L = [\log(T/2)]$, $L \le [T/4]$, $L = o(T)$, $L \to +\infty$, as $T \to +\infty$.

It is clear that for large T, one has with the help of Lemma (2), either $[S_{L,L}] = \mathcal{O}(T)$ if $\mu_1 \neq 0$, or $[S_{L,L}] = o(T)$ and $T^{-\beta}[S_{L,L}] \to +\infty$ for any $\beta < 1$ if $\mu_1 = 0$. In both cases, this leads to $[S_{L,L}] \ge [\log(T/2)] = L$ for T large enough, yielding hence

$$\hat{m}_1 = max\{L, [S_{L,L}]\} = [S_{L,L}], \ \hat{m}_1 + \hat{m}_2 = [T/2].$$

Henceforth, behaviors of $\hat{m}_1$ and $\hat{m}_2$ are readily described by Lemma (1) and Lemma (2) for large sample size T.

Similarly and by construction, $[S_{\hat{m}_1,\hat{m}_2}]$ behaves like twice $[S_{L,L}]$, *i.e.* $2\hat{m}_1$, for large T, in such a way that one obtains

$$m_1 = \max\{\hat{m}_1, [S_{\hat{m}_1,\hat{m}_2}]\} = \max\{[S_{L,L}], [S_{\hat{m}_1,\hat{m}_2}]\} = [S_{\hat{m}_1,\hat{m}_2}]$$

for T large. It follows by Lemma (1) that for any $\alpha \in [0, 1/2[$,

$$\frac{S_{\hat{m}_1,\hat{m}_2}}{T} = \frac{\tilde{\sigma}_2\sqrt{\tilde{\mu}_1^2 + \frac{\tilde{\sigma}_1^2}{T}}}{\tilde{\sigma}_2\sqrt{\tilde{\mu}_1^2 + \frac{\tilde{\sigma}_1^2}{T}} + \tilde{\sigma}_1\sqrt{\tilde{\mu}_2^2 + \frac{\tilde{\sigma}_2^2}{T}}},$$

$$= \frac{\sigma_2\sqrt{\mu_1^2 + \frac{\sigma_1^2}{T}}}{\sigma_2\sqrt{\mu_1^2 + \frac{\sigma_1^2}{T}} + \sigma_1\sqrt{\mu_2^2 + \frac{\sigma_2^2}{T}}} + o\left(\frac{1}{\hat{m}_1^\alpha}\right),$$

so that by Lemma (2), one has either $[S_{\hat{m}_1,\hat{m}_2}] = \mathcal{O}(T)$ if $\alpha_1 \neq 0$, or $[S_{\hat{m}_1,\hat{m}_2}] = o(T)$ and $T^{-\beta}[S_{\hat{m}_1,\hat{m}_2}] \to +\infty$ for any $\beta < 1$ if $\mu_1 = 0$, while $m_2 = T - m_1$ stills always an $\mathcal{O}(T)$ since 2 is assumed non zero. Thus m_1 is at most an $\mathcal{O}(m_2)$ when $\mu_1 \neq 0$ and otherwise $m_1 = o(m_2)$ for large samples. Hence, for any $\alpha \in [0, 1/2[$, Lemma (1) gives in one hand,

$$\frac{m_1}{T}\sigma_1\sqrt{\mu_2^2 + \frac{\sigma_2^2}{T}} - \frac{m_2}{T}\sigma_2\sqrt{\mu_1^2 + \frac{\sigma_1^2}{T}} = o\left(\frac{1}{m_1^\alpha}\right) + o\left(\frac{1}{m_2^\alpha}\right) = o\left(\frac{1}{m_1^\alpha}\right). \tag{7.7}$$

On the other hand, if $\mu_1 \neq 0$, m_1, m_2 being of same order of $T \to +\infty$, thus

$$o\left(\frac{1}{m_1^\alpha}\right) = o\left(\frac{1}{T^\alpha}\right), \frac{T}{m_1 m_2} = \mathcal{O}\left(\frac{1}{T}\right), \tag{7.8}$$

and if $\mu_1 = 0$, $\mu_1 = o(T)$ implies $m_2 = T \to m_1 = \mathcal{O}(T)$ and by Lemma (2), for any β, $0 \leq \beta < 1$, $T^{-\beta} m_1 \to +\infty$, thus

$$\frac{1}{m_1} = o\left(\frac{1}{T^\beta}\right)$$

and one obtain

$$\frac{1}{m_1^\alpha} = o\left(\frac{1}{T^{\alpha\beta}}\right), \frac{T}{m_1 m_2} = o\left(\frac{1}{T^\beta}\right), \tag{7.9}$$

since $T/m_2 = \mathcal{O}(1)$, for large T and for any β such that $0 \leq \beta < 1$. We obtain henceforth, using (7.7), (7.8) and (7.9),

$$E(m_1, m_2) = \frac{T^2\left(\frac{m_1}{T}\sigma_1\sqrt{\mu_2^2 + \frac{\sigma_2^2}{T}} - \frac{m_2}{T}\sigma_2\sqrt{\mu_1^2 + \frac{\sigma_1^2}{T}}\right)^2}{T m_1 m_2}.$$

$$= \begin{cases} o\left(\frac{1}{T^{2\alpha+1}}\right), \text{ if } \mu_1 \neq 0, \\ o\left(\frac{1}{T^{(2\alpha+1)\beta}}\right), \text{ if } \mu_1 = 0, \end{cases}$$

with probability one, for any $\alpha \in [0, 1/2[$, and all $\beta \in [0, 1[$. Thus in all cases of $\{\mu_1, \mu_2 \neq 0\}$, one concludes that

$$E(m_1, m_2) \leq o\left(\frac{1}{T^{\gamma}}\right)$$

with probability one as $T \to +\infty$, for all $\gamma \in [0, 2[$, $\gamma = 2\alpha + 1$ if $\mu_1 \neq 0$ and $\gamma = (2\alpha + 1)\beta$ if not, which ends the proof.

References

Benkamra, Z., Terbeche, M., and Tlemcani, M. (2013). Bayesian sequential estimation of the reliability of a parallel-series system. *Appl. Math. Comput.*, 219(23): 10842-10852.

Benkamra, Z., Terbeche, M., and Tlemcani, M. (2015). Nearly second order three-stage design for estimating a product of several Bernoulli proportions. *J. Statist. Plann. Inference,* 167: 90-101.

Berger, J. O. and Bernardo, J.-M. (1989). Estimating a product of means. Bayesian analysis with reference priors. *J. Amer. Statist. Assoc.*, 84(405): 200-207.

Chaturvedi, A., Bapat, S. R., and Joshi, N. (2020). Second-order approximations for a multivariate analog of Behrens-Fisher problem through three-stage procedure. *Comm. Statist. Theory Methods*, 49(14): 3466-3480.

Hardwick, J. and Stout, Q. F. (2002). Optimal few-stage designs. *J. Statist. Plann. Inference*, 104(1): 121-145.

Kim, H.-J. (2006). On Bayesian estimation of the product of Poisson rates with application to reliability. *Comm. Statist. Simulation Comput.*, 35(1): 47-59.

Liu, W. and Wang, N. (2007). A note on Hall's triple sampling procedure: a multiple sample second order sequential analogue of the Behrens-Fisher problem. *J. Statist. Plann. Inference*, 137(1): 331-343.

Maji, R., Singh, G., and Bandyopadhyay, A. (2018). Effective estimation of ratio and product of two population means in presence of random nonresponse in successive sampling. *Journal of Statistics Applications & Probability*, 7(2): 363-378.

Page, C. (1987). Sequential designs for estimating the product of parameters. Sequential Anal., 6(4): 351-371.

Page, C. F. (1990). Allocation proportional to coecients of variation when estimating the product of parameters. *J. Amer. Statist. Assoc.*, 85(412): 1134-1139.

Raubenheimer, L. and van der Merwe, A. J. (2012). Bayesian inferences on nonlinear functions of Poisson rates. *South African Statist. J.*, 46(2): 299-325.

Rekab, K. and Wu, W. (2015). Second-order test allocation for estimating reliability of series systems with two components. *JPSS J. Probab. Stat. Sci.*, 13(2): 249-262.

Robert, C. P. and Casella, G. (2004). Monte Carlo Statistical Methods. Springer Texts in Statistics, 2nd Edition. Springer New York, NY.

Singh, H. P. and Karpe, N. (2009). On the estimation of ratio and product of two population means using supplementary information in presence. Statistica, 69(1): 27-47.

Southwood, T. R. E. (1978). Ecological methods. With particular reference to the study of insect populations. Springer Netherlands, 14 d–ec. 1978-524 pages.

Stout, W. F. (1974). *Almost sure convergence*. Probability and Mathematical Statistics, Vol. 24. Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London.

Sun, D. and Ye, K. (1999). Reference priors for a product of normal means when variances are unknown. *Canad. J. Statist.*, 27(1): 97-103.

Sun, D. C. and Ye, K. (1995). Reference prior Bayesian analysis for normal mean products. *J. Amer. Statist. Assoc.*, 90(430): 589-597.

Woodroofe, M. and Hardwick, J. (1990). Sequential allocation for an estimation problem with ethical costs. *Ann. Statist.*, 18(3): 1358-1377.

Yfantis, E. A. and Flatman, G. T. (1991). On condence intervals for the product of three means of three normally distributed populations. *Journal of Chemometrics*, 5.

Zheng, S., Sriram, T. N., and Seila, A. F. (1998). Sequential xed-width confidence interval for the product of two means. *Ann. Inst. Statist. Math.*, 50(1): 119-145.